Noise in Reconstructed Images in Tomography
Parallel, Fan and Cone Beam Projections

Harish P. Hiriyanniah, Wesley E. Snyder and Griff L. Bilbro
Center for Communications and Signal Processing
North Carolina State University
Box 7914, Raleigh NC 27695-2914.

Abstract
A principal source of noise in X-ray and Emission Computer Tomography is at the detectors that measure the projection of the object. This noise source can be suitably characterized by Poisson statistics, given the nature of the photon emissions. However, due to the large photon count at the detector and the fact that the photon source is a point source, we may approximate the noise by additive, signal independent white Gaussian statistics by ignoring the negligible effects of scattering and diffraction of the source beam. Such an approximation for the observed noise is justified and mathematically prudent[2,4,5].

The back projection filter that is employed significantly alters power spectral density of the noise, such that the backprojected noise at each reconstructed point is no longer white. However, to effectively restore the reconstructed image using maximum a posteriori techniques via mean field annealing[1], a reasonable estimate of the autocorrelation of the back-projected noise is necessary. In the succeeding sections, we will derive the auto-correlation and power spectra of the reconstructed noise for parallel, fan and cone beam projections and also show some preliminary results of MFA restorations[1].

1 Noise in Parallel Beam Reconstruction
Let us first consider the 2-D slice reconstruction from 1-D projections. If \( P(t) \) is the actual projection of a slice \( f(x, y) \) at an angle \( \theta \) with \( t = x \cos \theta + y \sin \theta \) (See Figure 1) , the measured noisy projection \( P^\eta(t) \) is given by

\[
P^\eta(t) = P(t) + \eta(t)
\]

where \( \eta(t) \) is a signal independent zero mean, white noise process. The autocorrelation of the noise process is

\[
R_\eta(t_1,t_2,\theta_1,\theta_2) = \delta(t_1 - t_2) \delta(\theta_1 - \theta_2).
\]

The 2-D object is recovered from the projections \( P(t) \); \( 0 \leq \theta < \pi \) through the filtered back projection equation[2,5] expressed as

\[
f(x,y) = \int_0^\pi Q_\theta(t) \, d\theta \ ; \ Q_\theta(t) = P(t) \ast h(t).
\]

Here \( h(t) \) is the back projection filter and \( \ast \) denotes convolution. Utilizing the linearity of the back projection operation, we obtain the noisy reconstruction from the noisy projection as follows.

\[
f^\eta(x,y) = f(x,y) + \hat{f}(x,y)
\]
where \( f'(x, y) \) is the noisy reconstruction and \( f(x, y) \) is the noisy component of the reconstruction. i.e., \( f(x, y) = f_0' \eta_0(t) * h(t) d\theta \). We would like to characterize the auto-correlation of \( f(x, y) \), denoted as \( R(x_1, y_1 : x_2, y_2) = E\{f(x_1, y_1)f(x_2, y_2)\} \).

We will show that \( R(x_1, y_1 : x_2, y_2) \) is space-invariant, and hence, the reconstructed noise \( f(x, y) \) is stationary, though not white. We have

\[
R(x_1, y_1 : x_2, y_2) = \int_0^\pi \int_0^\pi E\{(\eta_0(t_1) * h(t_1))(\eta_0(t_2) * h(t_2))\} \, d\theta_1 \, d\theta_2
\]

\[
= S_0 \int_0^\pi H(t_2 - t_1) \, d\theta.
\]

Here, \( H(t) = h(t) * h(t) = \mathcal{F}^{-1}\{|H(\omega)|^2\}; H(\omega) = \mathcal{F}\{h(t)\} \). \( \mathcal{F} \) is the Fourier transform operator. It should also be noted that \( t_2 - t_1 = (x_2 - x_1) \cos \theta + (y_2 - y_1) \sin \theta \). Hence \( R(x_1, y_1 : x_2, y_2) \) is space invariant. Furthermore, expressing \( R(\alpha, \beta) \) in its polar form \( R(r, \phi) \) where \( \alpha = r \cos \phi \) and \( \beta = r \sin \phi \), we may write

\[
R(r, \phi) = S_0 \int_0^\pi \int_{-\infty}^{\infty} |H(\omega)|^2 e^{2\pi i r \omega \cos(\phi - \phi') \omega} \, d\omega \, d\theta = S_0 \int_0^\pi |H(\omega)|^2 J_0(2\pi r \omega) \, d\omega.
\]

The above equation illustrates the zero-order Hankel transform relationship between the auto-correlation function and the spectral density. \( J_0(x) \) is the zero-order Bessel function of the first kind. We also observe that \( R(r, \phi) \) is radially symmetric. Hence, the noise correlation between any two points depends only on their distance and not on their orientation with respect to any preferred axis.

The above analysis is easily extended to parallel beam 3-D reconstruction from 2-D projections. The 3-D auto-correlation function \( R(\alpha, \beta, \gamma) \) is then given by

\[
R(\alpha, \beta, \gamma) = S_0 \int_0^\pi H(t_2 - t_1, s_2 - s_1) \, d\theta \quad ; \quad H(t, s) = h(t, s) * * h(t, s)
\]

where \( ** \) denotes two dimensional convolution and \( h(t, s) \) is the 2-D backprojection filter. If \( h(t, s) \) is a circularly symmetric function (i.e., a function of \( \sqrt{t^2 + s^2} \)), then it is easily seen that \( R(\alpha, \beta, \phi) \) (expressed in spherical coordinates) is spherically symmetric, and depends only on \( r \), the absolute distance between two points \( (x_1, y_1, z_0) \) and \( (x_2, y_2, z_2) \). Indeed, without any loss of generality, we may employ such circularly symmetric filters [3]. If a smoothing window has to be used to form a composite back-projection filter to reduce the artifacts in the reconstruction, such a window must also be circularly symmetric.

2 Noise in Fan Beam Reconstruction

The fan beam projection process can be derived from an equivalent parallel beam projection. Instead of a mobile and collimated source-detector pair capable of both translation and rotation to obtain a parallel beam, we assume a point source irradiating the object with a fan beam, projecting it onto a detector array. Each ray in the beam is angularly displaced from the central ray of the fan beam (See Figure 2). Depending on the geometry of the detector array, we have two cases of fan beam projections.
• Equi-angular beam - The detector array is placed in a circular arc, with each detector element placed at equi-angular intervals. The fan beam angle is wide enough to completely irradiate the object at any source angle $\beta$.

• Equi-distant detector beam - The detector array is made up of equally spaced collinear elements. The equi-distant detector elements imply that the angles by each detector subtended at the source are not equal.

While the basic principle of reconstruction is the same in both the cases, the geometry of the detectors impose slightly different weighting functions during back-projection. Mathematically, this significantly alters the way the noise in the projections affects the reconstruction.

2.1 Equi-angular Fan Beam

The equiangular fan beam projection is illustrated in Figure 2. From the figure it is obvious that $\theta = \beta + \gamma$ and $t = D \sin \gamma$. Using the parallel beam back-projection equation expressed in polar coordinates, we have

$$f(r, \phi) = \frac{1}{2} \int_0^{2\pi} \int_{-\gamma}^{\gamma} P_\theta(t) h(r \cos(\theta - \phi) - t) \, dt \, d\theta. \quad (9)$$

where $h(t)$ is the back-projection filter. Substituting for $t$ and $\theta$, we obtain

$$f(r, \phi) = \frac{1}{2} \int_0^{2\pi} \int_{-\gamma}^{\gamma} P_{D+\gamma}(D \sin \gamma) h \{ r \cos(\beta + \gamma - \phi) - D \sin \gamma \} D \cos \gamma \, d\gamma \, d\beta.$$

The projection $P_{D+\gamma}(D \sin \gamma)$ is represented by $R_\theta(\gamma)$ in Figure 2. Assuming that the noise at the detector is white and Gaussian, the projection is given by

$$P_{D+\gamma}^n(D \sin \gamma) = P_{D+\gamma}(D \sin \gamma) + \eta_{D+\gamma}(D \sin \gamma). \quad (10)$$

Using Cartesian coordinates we obtain the back-projected noise $\hat{f}(x, y)$ as

$$\hat{f} = \frac{1}{2} \int_0^{2\pi} \int_{-\gamma}^{\gamma} \eta_{D+\gamma}(D \sin \gamma) h \{ x \cos(\beta + \gamma) + y \sin(\beta + \gamma) - D \sin \gamma \} D \cos \gamma \, d\gamma \, d\beta.$$

The auto-correlation function $\mathcal{R}(x_1, y_1 : x_2, y_2)$ is given by

$$\mathcal{R}(x_1, y_1 : x_2, y_2) = \frac{1}{4} \int_0^{2\pi} \int_{-\gamma}^{\gamma} \int_0^{2\pi} \int_{-\gamma}^{\gamma} E \{ \eta_{D+\gamma}(D \sin \gamma_1) \eta_{D+\gamma}(D \sin \gamma_2) \}
\times
\{ h \{ x_1 \cos(\beta_1 + \gamma_1) + y_1 \sin(\beta_1 + \gamma_1) - D \sin \gamma_1 \}
+ \{ h \{ x_2 \cos(\beta_2 + \gamma_2) + y_2 \sin(\beta_2 + \gamma_2) - D \sin \gamma_2 \}
- \{ h \}^2 \cos \gamma_1 \cos \gamma_2 \} \, d\gamma_1 \, d\beta_1 \, d\gamma_2 \, d\beta_2. \quad (11)$$

The autocorrelation of noise may be denoted, without loss of generality, as follows:

$$E \{ \eta_{D+\gamma}(D \sin \gamma_1) \eta_{D+\gamma}(D \sin \gamma_2) \} = \frac{S_n}{D \cos \gamma} \delta(\beta_1 + \gamma_1 - \beta_2 - \gamma_2) \delta(D \sin \gamma_1 - \sin \gamma_2)$$

$$= S_n \delta(\beta_1 - \beta_2) \delta(\gamma_1 - \gamma_2). \quad (12)$$
This is true since $\gamma_m$ is, in general, less than $\pi/2$. We therefore have

$$ R = \int_{-\gamma_m}^{\gamma_m} \int_{0}^{\gamma} h(m \cos \gamma + m \sin \gamma) h(p \cos \gamma + q \sin \gamma) \cos \gamma d\gamma d\beta, $$

where

$$ m = x_2 \cos \beta + y_2 \sin \beta; \quad n = -x_2 \sin \beta + y_2 \cos \beta - D $$

$$ p = x_1 \cos \beta + y_1 \sin \beta; \quad q = -x_1 \sin \beta + y_1 \cos \beta - D. \quad (13) $$

Substituting for $\beta$ and $\gamma$ in the previous equation, we obtain

$$ R = \int_{0}^{\gamma} \int_{-\gamma_m}^{\gamma_m} h(x_1 \cos \theta + y_1 \sin \theta - t_1) h(x_2 \cos \theta + y_2 \sin \theta - t_2) dt_1 dt_2 \quad (14) $$

The equation above proves that $R$ is space invariant. Simplifying the inner integral, while noting that $t_2 - t_1 = (x_2 - x_1) \cos \theta + (y_2 - y_1) \sin \theta$, we obtain

$$ R(x_2 - x_1, y_2 - y_1) = \int_{0}^{\gamma} H(t_2 - t_1) dt. \quad (15) $$

As in the parallel beam case, $H(t) = h(t) \star h(t)$. Using the same analysis, it can be easily shown that $R$ is radially symmetric. $h(t)$ can be obtained from the fan beam back projection filter as follows [2,5].

$$ h(x, y, \beta, \gamma) = \frac{1}{L(x, y, \beta) \sin^2(\gamma - \gamma') \phi(\gamma)} \quad (16) $$

$$ L^2(x, y, \beta) = [D + r \sin(\beta - \phi)]^2 + [r \cos(\beta - \phi)]^2 \quad (17) $$

$$ \gamma' = \tan^{-1} \frac{y \cos \beta + y \sin \beta}{D + x \sin \beta - y \cos \beta} \quad (18) $$

$(r, \phi)$ is the polar representation of $(x, y)$. Thus, using Equations 16-18 and Equation 15, the auto-correlation function is estimated. Since $R$ is circularly symmetric, its basic form can be obtained by calculating the correlation along the $x$ axis, by setting $y_2 = y_1 = 0$ in Eqn 15.

### 2.2 Equi-distant Collinear Detector Fan Beam

The analysis for this type of fan beam projection is similar to the equi-angular case. From Figure 3, we note that $t = s \cos \gamma$ and $\theta = \beta + \gamma$. The detectors are collinear and hence, the projection $P_{\beta + \gamma}(s \cos \gamma)$ is similar to parallel beam projection. Substituting for $\gamma$ we obtain

$$ t = \frac{sD}{\sqrt{D^2 + s^2}}; \quad \theta = \beta + \tan^{-1} \frac{s}{D}. \quad (19) $$

The projection $P_{\beta}(t)$ can thus be represented as $R_{\beta}(s)$. Using these results in the parallel beam back-projection equation, we have

$$ f(x, y) = \frac{1}{2} \int_{-\gamma_m}^{\gamma_m} \int_{0}^{\gamma} R_{\beta}(s) \frac{sD}{\sqrt{D^2 + s^2}} \left( \frac{D^3}{(D^2 + s^2)^{3/2}} \right) ds d\beta $$

$$ h \left[ x \cos(\beta + \gamma) + y \sin(\beta + \gamma) - \frac{sD}{\sqrt{D^2 + s^2}} \right] \left( \frac{D^3}{(D^2 + s^2)^{3/2}} \right) ds d\beta $$
The noise process $\eta(t)$ can be represented as $\eta(t)$ with its auto-correlation

$$R_n(s, s', \beta, \beta') = S_n \delta(s - s') \delta(\beta - \beta') \left( \frac{D'^2 + s'^2}{D^2} \right)^{3/2}.$$ (20)

By taking the appropriate expectation of the reconstructed noise as demonstrated in the previous section, the autocorrelation of the reconstructed noise, after substituting for $s$ and $\beta$, can be found to be

$$R(x_1, y_1 : x_2, y_2) = S_n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1 - t) h(t_2 - t) \, dt \, d\theta.$$ (21)

Simplifying the inner integral, we obtain $R(x_2 - x_1, y_2 - y_1) = S_n \int_{-\infty}^{\infty} H(t_2 - t) \, d\theta$. As before, it is easy to show that $R$ is space invariant radially symmetric. If $g(s)$ is the actual back-projection filter that is applied the projection data prior to back-projection, then $h(x, y, s, \beta)$ is related to $g(s)$ as[2,5]

$$h(x, y, s, \beta) = \frac{D^2 + (s' - s)^2}{2U^2D^2} g(s)$$ (22)

where

$$s' = \frac{D \cos \beta + y \sin \beta}{D \cos \beta + x \sin \beta} \quad U(x, y, \beta) = \frac{D \sin \beta - y \cos \beta}{D} \quad (23)$$

### 3 Noise in Cone Beam Reconstruction

Most cone beam reconstructions are an extension of collinear detector fan beam projection case. Here, the cone beam is projected through the three dimensional object on to a planar detector surface, with the detector elements arranged on a square grid. Using the reconstruction analysis in [2,3], each row of the projection is treated as a tilted fan beam, tilted such that the beam hits the detector plane at an altitude $\kappa$ above the reference horizontal axis of the detector plane. The analysis of such a reconstruction method follows very closely that of the collinear detector fan beam reconstruction, except that a correction factor is applied to the source to origin distance $D$. This distance is replaced by $D_\psi$, where

$$D_\psi^2 = D^2 + \psi^2.$$ (24)

Here $\psi$ is the height of the tilted fan above the center of the plane of rotation. Applying this correction factor the auto-correlation function can be easily determined as before. Once again, we postulate (without proof at this time) that a space-invariant form of the auto-correlation function can be obtained. It is easy to visualize that $R$ will be spherically symmetric if circularly symmetric backprojection filters are used.
4 Results

Figure 4 shows six pictures of the restoration of a reconstructed ellipse. Clockwise from top left, the first picture shows an ellipse of constant intensity. The second picture shows the reconstructed ellipse from its noisy projection. White noise has been added to the projection before performing filtered back-projection. The third picture shows the restoration of the noisy reconstruction assuming that the noise was white [1]. The next picture is the difference image with respect to the original, clean ellipse. Note that the outline of the ellipse has shifted by one pixel during the forward projection and filtered back-projection processes. The next picture shows the restoration using the colored noise model. The power spectrum of the noise was estimated from the back-projection filter used. The last picture is the difference image with respect to the original image. Note that the colored noise restoration is marginally better in restoring the edges and smoothing the outliers.

Figure 5 shows three pictures. Clockwise from top left, the first picture is a real reconstructed image of a GaAs transistor. The second picture shows the reconstruction assuming a white noise model. The last picture shows the reconstruction assuming a colored noise model. The back-projection filter had a Shepp-Logan kernel. Again, we notice that the restoration using colored noise model is marginally better qualitatively, although not much difference was noticed in the statistics of the difference images.

References


Figure 1: Parallel Beam Projection

Figure 2: Fan Beam Projection — Equi-angular rays

Figure 3: Fan Beam Projection — Equi-distant collinear detectors
Figure 4: Restoration of a synthetic ellipse. See Results for explanation.

Figure 5: Restoration of a GaAs transistor. See Results for explanation.